The answers to the exercises should be returned before **January 31, 2008**.

The answers should be sent as email to the teacher (ssarkka@lce.hut.fi) in PDF form. When sending the email,please add "S-114.4220" or "1144220" to subject. The answers can also be returned on paper to the teacher.

## Exercise 1. (Steady State LQR)

Assume that we want to regulate the position of spring $x(t)$, which satisfies the differential equation

$$\ddot{x} + \dot{x} + x = u(t) \tag{1}$$

to the origin. $u(t)$ is the control force and the cost functional to be minimized by the control is

$$J(u(\cdot)) = \frac{1}{2} \int_0^\infty \left[ x^2(t) + \dot{x}^2(t) + u^2(t) \right] dt \tag{2}$$

and the initial conditions are $x(0) = x_0$, $dx/dt(0) = 0$.

**(A)** Write this optimal control problem as canonical steady state linear quadratic regulation problem with two dimensional state (see, e.g., the lecture notes).

**(B)** Write down the corresponding algebraic Riccati equation (ARE) and the optimal gain in terms of the ARE solution.

**(C)** Use Matlab's (or Octave's) LQR function for solving the optimal gain and the ARE solution. Check that the LQR function's ARE solution indeed solves the ARE in (B) and the optimal gain is given by the equation you gave above.

## Exercise 2. (Sequential Decision Making Under Uncertainty)

The following exercise is based on the exercise 22.1 of the book *Gelman, Carlin, Stern, Rubin: Bayesian Data Analysis*.

Oscar has lost his dog; there is a 70% probability it is in forest A and a 30% chance it is in forest B. If the dog is in forest A and Oscar looks there for a day, he has a 50% chance of finding the dog. If the dog is in the forest B and Oscar looks there for a day, he has an 80% chance of finding the dog.

Assume that Oscar can spend arbitrary number of days such that on one day he searches the dog in a one forest for the whole day. Also assume that the dog doesn't move from forest to another. Define the following:

- The state at step $k$ is two dimensional $\mathbf{x}_k = (x_{k,1}\; x_{k,2})^T$ and has the following possible values:

$$\begin{aligned} x_{k,1} &= \begin{cases} Y = \text{Dog is found at or before the current step} \\ N = \text{Dog has not been found yet} \end{cases} \\ x_{k,2} &= \begin{cases} A = \text{Dog is in forest A} \\ B = \text{Dog is in forest B} \end{cases} \end{aligned} \tag{3}$$

- The control signal is

$$u_k = \begin{cases} \mathrm{S_A} = \text{Search from forest A} \\ \mathrm{S_B} = \text{Search from forest B} \end{cases} \tag{4}$$

- A measurement signal, which is what we observe, is defined as

$$\begin{aligned} y_k &= x_{k,1} \\ &= \begin{cases} Y = \text{Dog is found at or before the current step} \\ N = \text{Dog has not been found yet} \end{cases} \end{aligned} \tag{5}$$

**(A)** Write down the probability distributions $P(\cdot)$ such that the model can be written in "state space" form

$$\begin{aligned} x_k &\sim P(x_k \,|\, x_{k-1}, u_k) \\ y_k &\sim P(y_k \,|\, x_k), \end{aligned} \tag{6}$$

with an initial distribution $x_0 \sim P(x_0)$. Write a computer program for evaluating these probabilities.

**(B)** At any time step $k$ we want to make a decision $u_k$, which maximizes the expected payoff conditionally to the already made decisions and observations:

$$E_x\left[R_k(u_k; x_1, \ldots, x_k) \,|\, u_1, \ldots, u_{k-1}, y_1, \ldots, y_{k-1}\right] \tag{7}$$

for a given $y_1, \ldots, y_T$, where

$$R_k(u_k; x_1, \ldots, x_k) = \begin{cases} 1 & , \quad \text{if } x_{k,1} = Y \\ 0 & , \quad \text{if } x_{k,1} = N \end{cases} \tag{8}$$

What is the optimal value $u_1$, which maximizes $E_x[R_1(u_1, x_1)]$, that is, the expected payoff for searching for the first day? What is the probability that the dog is still lost after the search?

**(C)** Assume Oscar made the rational decision and the dog is still at lost (and is still in the same forest as yesterday). Where should he search for the dog on the second day? What is the probability that the dog is still lost at the end of the second day?

**(D)** Write a compute program that computes at arbitrary time step $k$ the optimal search decision $u_k$ given that the dog has not yet been found, that is, $y_1 = N, \ldots, y_{k-1} = N$. What are the first 10 optimal decisions $u_1, \ldots, u_{10}$? If dog still isn't found, in which forest the dog almost certainly is in?