

Introduction to Classical, Optimal and Stochastic Control

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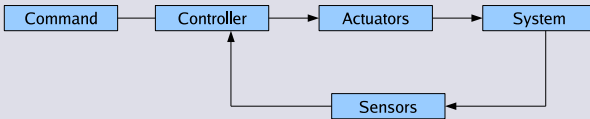
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Principles and History of Control

- Control theory studies how a **physical system** can be steered to a given **goal** by applying a **control signal**
- **Examples of applications** are Cruise control, Autopilot, Spacecraft control, Chemical process control, Nuclear reactor control, Robot control and Disk drive control
- Dates back to 1700's and 1800's, but most of the **classical theory** was developed in beginning of 1900's.
- The "modern" **state space methods, optimal control theory, stochastic control theory** and many numerical algorithms were developed in 1960's.

Building Blocks and Approaches of Control



- The **physical system** (plant) is monitored with **sensors** and the control signal is generated with **actuators**
- In **classical control**, a controller is **heuristically** designed such that it forces **error** with respect to reference **to zero**
- In **optimal control**, the controller is selected to **minimize a cost functional**
- In **stochastic control**, the **system model and parameters** are considered **uncertain** and the **state** is only **indirectly measured**

Mathematical Foundations of Control Theory

- The **physical system** is a set of **differential or difference equations**, deterministic or stochastic
- The **control signal** is an **unknown function**, which is to be determined
- In **classical control**, linear control laws are derived by heuristic methods and their stability is checked using **Laplace, Fourier and Z-transforms**
- In **optimal control** the control signal is solved using **calculus of variations** and its generalizations
- In **stochastic control**, **stochastic generalizations of optimal control results** are used for determining the control signals.
- Also in **stochastic control**, uncertainties and noises in the system and sensors are treated according to **Bayesian probability theory**

Transfer Function Models: Theory [1/2]

- The **classical theory** considers **linear time-invariant (LTI)** systems with **single input and single output (SISO)**
- In continuous-time, **constant coefficient linear differential equations** for signal $y(t)$ with input $u(t)$

$$\sum_{i=0}^n a_i \frac{d^i y(t)}{dt^i} = \sum_{j=0}^m b_j \frac{d^j u(t)}{dt^j}$$

- Analysis is based on **Laplace transforms** of the differential equations and their **transfer functions**

$$G(s) = \frac{Y(s)}{U(s)} = \frac{\sum_{j=0}^m b_j s^j}{\sum_{i=0}^n a_i s^i}$$

Transfer Function Models: Theory [2/2]

- In discrete-time, **constant coefficient linear difference equations**

$$\sum_{i=0}^n a_i y[t - i] = \sum_{j=0}^m b_j u[t - j]$$

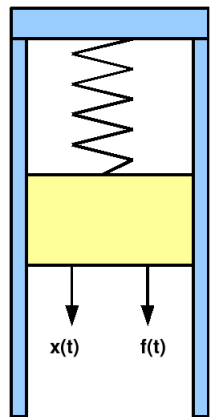
- In terms of **z-transform** the **transfer function** is

$$G(z) = \frac{Y(z)}{U(z)} = \frac{\sum_{j=0}^m b_j z^{-j}}{\sum_{i=0}^n a_i z^{-i}}$$

- Input to the system (continuous or discrete time) is the **reference/command signal**
- Analysis is based on **impulse and step response** to the input, and the **steady state response**

Transfer Function Models: Example [1/3]

Example



- Spring force

$$F_j = -by$$

- Friction force

$$F_k = -kdy/dt$$

- External force

$$F_f = f(t)$$

- Newton's law:

$$M \frac{d^2y}{dt^2} = F_k + F_j + F_f$$

Transfer Function Models: Example [2/3]

- If we assume $M=k=1$ for simplicity:

$$d^2y/dt^2 + b dy/dt + y = f(t)$$

- Take **Laplace transform** of both sides:

$$s^2 Y(s) + b s Y(s) + Y(s) = F(s)$$

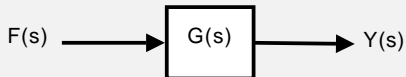
- This can be written in form

$$Y(s) = \underbrace{\left[\frac{1}{s^2 + b s + 1} \right]}_{G(s)} F(s)$$

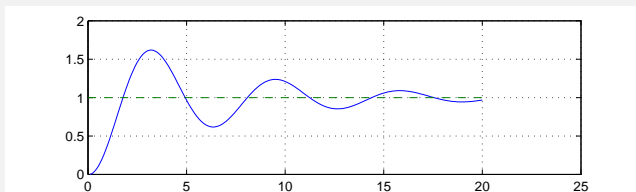
- Here $G(s)$ is the **transfer function** from $f(t)$ to $y(t)$

Transfer Function Models: Example [3/3]

- As a **block diagram** the system is



- If we apply a **constant force** $f(t) = 1$, we get the following **step response**:

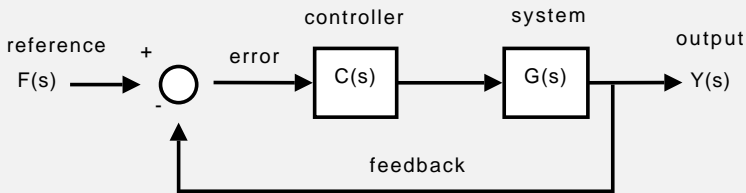


Classical Control Problem

- **Classical control problem:** *Design a control signal $u(t)$ such that $y(t)$ reaches its steady state as fast/well/efficiently as possible.*
- Force $f(t)$ is the **reference/command signal** and we want $y(t)$ to follow it
- Or might want to **“measure” the force $f(t)$** by monitoring the position $y(t)$ of spring (think of accelerometer)
- How to select the controller?
- How could we get rid of the oscillations and get faster response?

Controller types: Theory

- The idea is to include **controller** $C(s)$ and **feedback** such that the **system** follows the given **reference signal** $F(s)$:



- The most known type of controller $C(s)$ is the **proportional-integral-derivative (PID)** controller

$$C(s) = K_p + K_i/s + K_d s$$

- In time-domain**, the PID-controller is

$$u(t) = K_p e(t) + K_i \int_0^t e(t) dt + K_d de/dt$$

Controller types: Example [1/3]

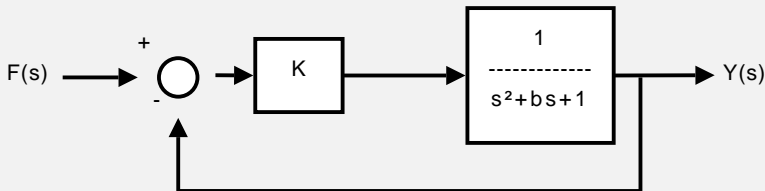
- Let's now add controller $C(s)$ to the spring system

$$Y(s) = \left[\frac{1}{s^2 + bs + 1} \right] F(s)$$

- Take a simple P-controller

$$C(s) = K$$

- The block diagram is the following:

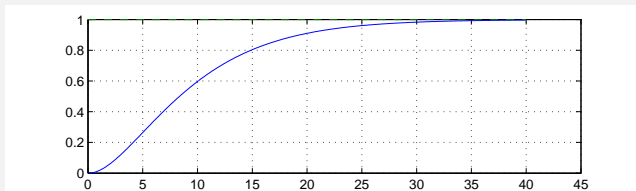


Controller types: Example [2/3]

- The closed loop system is now

$$Y(s) = \left[\frac{K}{s^2 + bs + 1 + K} \right] F(s)$$

- We may select K such that the system is **Critically damped**, that is, the fastest response before oscillations occur
- Applying a **constant force** $f(t) = 1$, gives the following response:



Controller types: Example [3/3]

- What this means **in physical system**? The equation of error $e(t) = y(t) - 1$ is now

$$u(t) = K e(t)$$

$$d^2 e/dt^2 + b de/dt + e = u(t)$$

- In terms of $y(t)$

$$u(t) = K(y(t) - 1)$$

$$d^2 y/dt^2 + b dy/dt + y = u(t)$$

- That is, we apply the following **control force**

$$u(t) = K(y(t) - 1)$$

Classical Design Methods

- Controller design is based on **trial and error** and **heuristic design** methods for selecting the controller parameters
- The principal design goal is **stability**: The system is stable when the **closed loop poles** are on the left-half of s -plane or inside the unit circle in z -plane
- Secondary criteria are for example, **settling time, overshoot and steady state error**. These can be analyzed graphically from **impulse, step and ramp responses** of the close loop system
- **Robustness** to noise and other disturbances can be analyzed using **frequency domain** methods
- It is also possible to design heuristic feedback laws of state space models and prove their stability, e.g., by **pole placement** or by **Lyapunov stability theory**

Problem Formulation

- The system is modeled as **state space model** with state vector $\mathbf{x}(t) = (x_1(t), \dots, x_n(t))$, and control vector $\mathbf{u}(t) = (u_1(t), \dots, u_d(t))$

$$d\mathbf{x}/dt = \mathbf{f}(\mathbf{x}, \mathbf{u}, t),$$

- The design criterion is to minimize **cost functional**

$$J(\mathbf{u}) = \phi(\mathbf{x}(T)) + \int_0^T L(\mathbf{x}, \mathbf{u}, t) dt.$$

- **Initial and final states** can be known, unknown or constrained
- **State** and **control** may be **constrained**
- The **final time** may be **known** or **unknown**

Euler-Lagrange and Hamiltonian Equations [1/2]

- The optimal control function $\mathbf{u}(t)$ can be derived using methods **calculus of variations**, namely the **Euler-Lagrange equations**
- The result is - we first define the **Hamiltonian**

$$H(\mathbf{x}, \mathbf{u}, \mathbf{p}, t) = L(\mathbf{x}, \mathbf{u}, t) + \mathbf{p}^T \mathbf{f}(\mathbf{x}, \mathbf{u}, t),$$

where $\mathbf{p}(t)$ is the **co-state** (= Lagrange multiplier function)

- Then the optimal control and trajectory can be solved from the **Hamiltonian equations**

$$d\mathbf{x}/dt = \partial H/\partial \mathbf{p}$$

$$d\mathbf{p}/dt = -\partial H/\partial \mathbf{x}$$

$$0 = \partial H/\partial \mathbf{u},$$

Euler-Lagrange and Hamiltonian Equations [2/2]

- Non-linear **boundary value problem (BVP)** for ordinary differential equations
- The **boundary conditions** depend on if the **final state** is **known or unknown** and if the **final time** is **known or unknown**
- Generally, **very hard to solve** even numerically
- The corresponding **discrete-time optimal control problem** is

$$\mathbf{x}_{k+1} = \mathbf{f}_k(\mathbf{x}_k, \mathbf{u}_k)$$
$$J(\mathbf{u}) = \sum_{k=0}^T L_k(\mathbf{x}_k, \mathbf{u}_k)$$

- Hamiltonian equations are almost the same, only in discrete time

Optimal Control: Example [1/2]

- Consider the spring problem:

$$d^2y/dt^2 + b dy/dt + y = u(t)$$

- If we define $x_1 = y$ and $x_2 = dy/dt$, this can be written as **state space model**

$$dx_1/dt = x_2$$

$$dx_2/dt = -x_1 - b x_2 + u$$

- Define the **cost functional** as

$$J(u) = \frac{1}{2} \int_0^T \left[\|x_1(t) - f(t)\|^2 + \|u(t)\|^2 \right] dt$$

Optimal Control: Example [2/2]

- The **Hamiltonian** is now

$$H(\mathbf{x}, u, \mathbf{p}, t) = \frac{1}{2} \|x_1(t) - f(t)\|^2 + \frac{1}{2} \|u(t)\|^2 \\ + p_1 x_2 + p_2 (-x_1 - b x_2 + u)$$

- The **Hamiltonian equations** now reduce to

$$dx_1/dt = x_2$$

$$dx_2/dt = -x_1 - b x_2 + u$$

$$dp_1/dt = -x_1 + f + p_2$$

$$dp_2/dt = -p_1 + b p_2$$

$$0 = u + p_2$$

- More useful form of solution given by **linear quadratic regulator** framework

Pontryagin Minimum Principle

- Euler-Lagrange and Hamiltonian equations apply when control is **not bounded**.
- If the control **is bounded**, e.g., $\|\mathbf{u}\| \leq 1$, then equations don't work - **Pontryagin maximum (or minimum) principle** must be used, where $\partial H/\partial \mathbf{u} = 0$ is replaced with

$$\mathbf{u}(t) = \min_{\mathbf{u}(t)} H(\mathbf{x}(t), \mathbf{u}(t), \mathbf{p}(t), t)$$

- Heuristically, generalizes the principle that the maximum of a function is either at point where **gradient is zero** or **at boundary**
- Especially useful in **minimum time** and **minimum fuel** problems, where there is no explicit control cost at all

Bang-Bang Control

- **Minimizing the time** with bounded control leads to **bang-bang-control**, where the control is always at the boundary, i.e., to **maximum effort** type of control such as

$$u(t) = \begin{cases} u_{\max} & , \quad \text{if } \varphi(\mathbf{p}(t)) < 0 \\ u_{\min} & , \quad \text{if } \varphi(\mathbf{p}(t)) > 0 \end{cases}$$

- **Minimizing the fuel** leads to control, where the control is always **at maximum, minimum or zero**
- Other **constrained problems** lead to solutions, which changes between bang-bang and ordinary control.
- The **discrete-time version** of the principle is analogous to continuous-time

Bang-Bang Control: Example [1/3]

- The equation of a forced **simple pendulum** is

$$d^2\theta/dt^2 = -g \sin(\theta) + u$$

- As state space model this is

$$dx_1/dt = x_2$$

$$dx_2/dt = -g \sin(x_1) + u$$

- We want to steer the pendulum from down-most position to up-most position in minimum time with **bounded control**
 $0 \leq |u(t)| \leq 1$
- The **minimum time cost function** is

$$J(u) = \int_0^T 1 dt$$

where T is unknown.

Bang-Bang Control: Example [2/3]

- The **Hamiltonian** is given as

$$H(\mathbf{x}, u, \mathbf{p}) = 1 + p_1 x_2 + p_2 (-g \sin(x_1) + u)$$

- The **Hamiltonian equations** are

$$dp_1/dt = p_2 g \cos(x_1)$$

$$dp_2/dt = -p_1$$

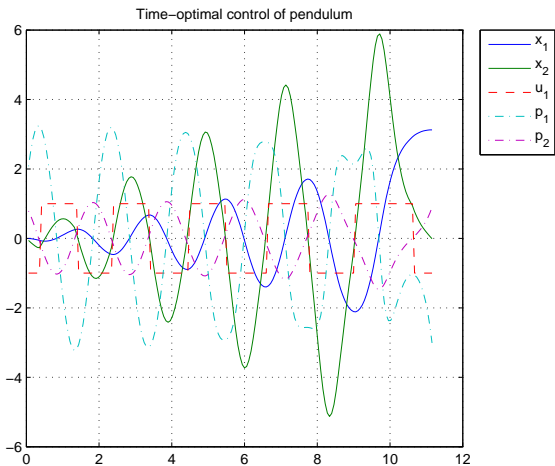
$$dx_1/dt = x_2$$

$$dx_2/dt = -g \sin(x_1) + u$$

- The Hamiltonian equation $\partial H/\partial u = 0$ **does not have** a nontrivial **solution** in the bounded domain.
- Instead, the **Pontryagin principle** gives:

$$u(t) = \begin{cases} 1 & , \text{ if } p_2(t) < 0 \\ -1 & , \text{ if } p_2(t) > 0 \end{cases}$$

Bang-Bang Control: Example [3/3]



Dynamic Programming [1/2]

- Bellman's **Principle of Optimality**: “Starting at any point on an optimal trajectory, the remaining trajectory is optimal for the corresponding problem initiated at that point.”
- In discrete-state case this is related to finding the **shortest path** between two vertexes **in a graph**.
- We may solve the following **discrete-time optimal control problem** using this principle

$$\mathbf{x}_{k+1} = \mathbf{f}_k(\mathbf{x}_k, \mathbf{u}_k)$$
$$J(\mathbf{u}) = \sum_{k=0}^T L_k(\mathbf{x}_k, \mathbf{u}_k)$$

Dynamic Programming [2/2]

- The **dynamic programming algorithm** is

1 Initialize: $V_{T+1}(\mathbf{x}_{T+1}) = 0$.

2 For $k = T, \dots, 0$ and for all \mathbf{x}_k compute

$$\mathbf{u}_k^*(\mathbf{x}_k) = \arg \min_{\mathbf{u}_k} \{L_k(\mathbf{x}_k, \mathbf{u}_k) + V_{k+1}(\mathbf{f}_k(\mathbf{x}_k, \mathbf{u}_k))\}$$

$$V_k(\mathbf{x}_k) = L_k(\mathbf{x}_k, \mathbf{u}_k^*) + V_{k+1}(\mathbf{f}_k(\mathbf{x}_k, \mathbf{u}_k^*))$$

3 For $k = 0, \dots, T - 1$, compute the optimal trajectory

$$\mathbf{x}_{k+1}^* = \mathbf{f}_k(\mathbf{x}_k^*, \mathbf{u}_k^*)$$

- $V_k(\mathbf{x}_k)$ is called the **optimal value function** or the **optimal cost-to-go function**
- Problem is the **curse of dimensionality**, because has to be solved for each \mathbf{x}_k separately

Hamilton-Jacobi-Bellman Equation

- The **Hamilton-Jacobi-Bellman (HJB)** equation is the continuous-time limit of the dynamic programming and can be applied to models of the form

$$d\mathbf{x}/dt = \mathbf{f}(\mathbf{x}, \mathbf{u}, t)$$

$$J(\mathbf{u}) = \phi(\mathbf{x}(T)) + \int_0^T L(\mathbf{x}, \mathbf{u}, t) dt.$$

- The HJB equation for the **optimal value function** $V(\mathbf{x}, t)$ is

$$\partial V / \partial t = - \min_{\mathbf{u}} H(\mathbf{x}, \mathbf{u}, \partial V / \partial \mathbf{x}, t)$$

$$= - \min_{\mathbf{u}} \left\{ L(\mathbf{x}, \mathbf{u}, t) + (\partial V / \partial \mathbf{x})^T \mathbf{f}(\mathbf{x}, \mathbf{u}, t) \right\}.$$

- **Non-linear partial differential equation** and very hard to solve even in simple cases
- Gives the optimal control in **state feedback form** $\mathbf{u}^*(\mathbf{x}(t), t)$

Linear Quadratic Regulator

- The **linear quadratic regulator (LQR)** model is

$$d\mathbf{x}(t)/dt = \mathbf{F}(t) \mathbf{x}(t) + \mathbf{B}(t) \mathbf{u}(t)$$

$$J(\mathbf{u}) = \frac{1}{2} \mathbf{x}(T)^T \Phi \mathbf{x}(T) + \frac{1}{2} \int_0^T (\mathbf{x}^T(t) \mathbf{Q}(t) \mathbf{x}(t) + \mathbf{u}^T(t) \mathbf{R}(t) \mathbf{u}(t)) dt,$$

- The **HJB equation** now has a **closed form solution**:

$$V(\mathbf{x}(t), t) = \frac{1}{2} \mathbf{x}^T(t) \mathbf{K}(t) \mathbf{x}(t)$$

$$\mathbf{u}(t) = -\mathbf{R}^{-1}(t) \mathbf{B}^T(t) \mathbf{K}(t) \mathbf{x}(t)$$

- The matrix $\mathbf{K}(t)$ solves the **Riccati differential equation**

$$d\mathbf{K}(t)/dt = -\mathbf{F}^T(t) \mathbf{K}(t) - \mathbf{K}(t) \mathbf{F}(t)$$

$$+ \mathbf{K}(t) \mathbf{B}(t) \mathbf{R}^{-1}(t) \mathbf{B}^T(t) \mathbf{K}(t) - \mathbf{Q}(t).$$

Steady State Linear Quadratic Regulator

- Consider **LTI linear quadratic regulator** model, where the dynamic model is LTI, the terminal time is at infinity and there is no terminal cost:

$$d\mathbf{x}(t)/dt = \mathbf{F} \mathbf{x}(t) + \mathbf{B} \mathbf{u}(t)$$

$$J(\mathbf{u}) = \frac{1}{2} \int_0^{\infty} (\mathbf{x}^T(t) \mathbf{Q} \mathbf{x}(t) + \mathbf{u}^T(t) \mathbf{R} \mathbf{u}(t)) dt,$$

- The solution is now **time invariant**:

$$V(\mathbf{x}(t), t) = \frac{1}{2} \mathbf{x}^T(t) \mathbf{K} \mathbf{x}(t)$$

$$\mathbf{u}(t) = -\mathbf{R}^{-1} \mathbf{B}^T \mathbf{K} \mathbf{x}(t)$$

- The constant matrix \mathbf{K} solves the **algebraic Riccati equation**

$$0 = -\mathbf{F}^T \mathbf{K} - \mathbf{K} \mathbf{F} + \mathbf{K} \mathbf{B} \mathbf{R}^{-1} \mathbf{B}^T \mathbf{K} - \mathbf{Q}.$$

Linear Quadratic Regulator: Example [1/3]

- The error equation of the spring control problem:

$$d^2 e / dt^2 + b de / dt + e = u(t)$$

- If we define $x_1 = e$ and $x_2 = de/dt$, can be written as **linear state space model**

$$\frac{d}{dt} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & -b \end{pmatrix}}_{\mathbf{F}} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\mathbf{B}} u$$

- Define the **cost functional** as

$$J(u) = \frac{1}{2} \int_0^{\infty} \left[\|e(t)\|^2 + c \|u(t)\|^2 \right] dt$$

Linear Quadratic Regulator: Example [2/3]

- The elements of 2×2 matrix \mathbf{K} can be solved from the algebraic equations

$$k_{21} + k_{12} + k_{12} k_{21}/c - 1 = 0$$

$$k_{22} - k_{11} + b k_{12} + k_{12} k_{22}/c = 0$$

$$-k_{11} + b k_{21} + k_{22} + k_{22} k_{21}/c = 0$$

$$-k_{12} + 2b k_{22} - k_{21} + k_{22}^2/c - 1 = 0$$

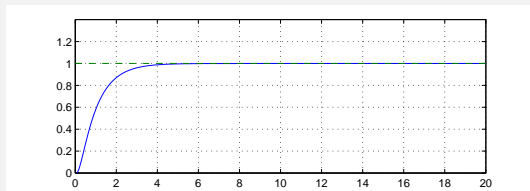
- The **optimal control** is of the form

$$u(t) = -\mathbf{B}^T \mathbf{K} \mathbf{x}(t) = -k_{21} e(t) - k_{22} de/dt$$

- This is a **proportional-derivative** (PD) controller

Linear Quadratic Regulator: Example [3/3]

- The **step response** of the PD controller is



- If we had augmented the system with **integral state**, we would have obtained a **PID controller**
- All **classical controllers** have a **counterpart** in LQR theory
- LQR framework provides a **systematic way** of selecting the **PID parameters** instead of heuristics

Problem Formulation - Perfectly Measured State

- The system contains **uncertainties in dynamics**, which are modeled with **noise process $\mathbf{w}(t)$** :

$$d\mathbf{x}/dt = \mathbf{f}(\mathbf{x}, \mathbf{u}, t) + \mathbf{L}(t) \mathbf{w}(t)$$

- The performance criterion is the **expected value of the cost function**:

$$J(\mathbf{u}) = \mathbb{E} \left[\phi(\mathbf{x}(T)) + \int_0^T L(\mathbf{x}, \mathbf{u}, t) dt \right].$$

- In **discrete-time** case the corresponding model is:

$$\mathbf{x}_{k+1} = \mathbf{f}_k(\mathbf{x}_k, \mathbf{u}_k, \mathbf{w}_k)$$

$$J(\mathbf{u}) = \mathbb{E} \left[\sum_{k=0}^T L_k(\mathbf{x}_k, \mathbf{u}_k) \right]$$

- The **state** is assumed to be **completely known** when the control is applied

Stochastic Dynamic Programming and HJB I

- The **discrete-time** solution is **almost the same** as in **deterministic case**, with a few additional expected values:

$$\mathbf{u}_k^*(\mathbf{x}_k) = \arg \min_{\mathbf{u}_k} E_{w_k} [L_k(\mathbf{x}_k, \mathbf{u}_k) + V_{k+1}(\mathbf{f}_k(\mathbf{x}_k, \mathbf{u}_k, \mathbf{w}_k))]$$

$$V_k(\mathbf{x}_k) = E_{w_k} [L_k(\mathbf{x}_k, \mathbf{u}_k^*) + V_{k+1}(\mathbf{f}_k(\mathbf{x}_k, \mathbf{u}_k^*, \mathbf{w}_k))]$$

- In **continuous-time** case there are no expected values, but the **stochastic HJB equation** contains an additional second order term:

$$\begin{aligned} \partial V / \partial t = - \min_{\mathbf{u}} \left\{ L(\mathbf{x}, \mathbf{u}, t) + (\partial V / \partial \mathbf{x})^T \mathbf{f}(\mathbf{x}, \mathbf{u}, t) \right. \\ \left. + \frac{1}{2} \operatorname{tr} \left[\left(\frac{\partial^2 V}{\partial \mathbf{x}^2} \right) \mathbf{L}(t) \mathbf{Q}_c(t) \mathbf{L}^T(t) \right] \right\} \end{aligned}$$

Stochastic Linear Quadratic Regulator

- The **stochastic linear quadratic regulator (LQR)** model is

$$d\mathbf{x}(t)/dt = \mathbf{F}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) + \mathbf{L}(t)\mathbf{w}(t)$$

$$J(\mathbf{u}) = \mathbb{E} \left[\frac{1}{2} \mathbf{x}(T)^T \boldsymbol{\Phi} \mathbf{x}(T) + \frac{1}{2} \int_0^T (\mathbf{x}^T(t) \mathbf{Q}(t) \mathbf{x}(t) + \mathbf{u}^T(t) \mathbf{R}(t) \mathbf{u}(t)) dt \right]$$

- The **optimal control** is exactly the **same as in LQR without the noise $\mathbf{w}(t)$** .
- The **optimal value function** is of the form

$$V(\mathbf{x}(t), t) = V_d(\mathbf{x}(t), t) + g(t)$$

where $V_d(\cdot)$ is the optimal value function of deterministic case.

Problem Formulation - Partially Measured State [1/2]

- In physical systems the **state** of the system is **never perfectly known**, but we observe it through some **sensors**
- Model for **sensor measurement** $\mathbf{y}(t)$ with noise/uncertainties $\mathbf{v}(t)$:

$$\mathbf{y}(t) = \mathbf{h}(\mathbf{x}(t), \mathbf{u}(t), t) + \mathbf{v}(t)$$

- **We do not observe** $\mathbf{x}(t)$, only $\mathbf{y}(t)$ and all our **decisions** must be based on the already observed $\mathbf{y}(t)$
- The **control decision** at time t is function of **measurements** obtained **before and at** time t
- If the **control** $\mathbf{u}(t)$ was **given**, the estimation of $\mathbf{x}(t)$ given also the dynamic model would be an **optimal filtering problem**.
- The situation is much more complicated - aside with optimal filtering, we want to also **optimize the expected value of cost function**.

Problem Formulation - Partially Measured State [2/2]

- The whole model in **continuous time**:

$$d\mathbf{x}/dt = \mathbf{f}(\mathbf{x}, \mathbf{u}, t) + \mathbf{L}(t) \mathbf{w}(t)$$

$$\mathbf{y} = \mathbf{h}(\mathbf{x}, \mathbf{u}, t) + \mathbf{v}(t)$$

$$J(\mathbf{u}) = \mathbb{E}_x \left[\phi(\mathbf{x}(T)) + \int_0^T L(\mathbf{x}, \mathbf{u}, t) dt \mid Y_t \cup Y_{[t,T]}^+ \right].$$

- In **discrete time**:

$$\mathbf{x}_{k+1} = \mathbf{f}_k(\mathbf{x}_k, \mathbf{u}_k, \mathbf{w}_k)$$

$$\mathbf{y}_k = \mathbf{h}_k(\mathbf{x}_k, \mathbf{u}_k, \mathbf{v}_k)$$

$$J(\mathbf{u}) = \sum_{k=0}^T \mathbb{E}_x \left[L_k(\mathbf{x}_k, \mathbf{u}_k) \mid \mathbf{y}_{1:k} \cup \mathbf{y}_{k+1:T}^+ \right]$$

Form of the Solution

- The **discrete-time** solution **resembles** the **perfectly measured** case, but the value function is now function of the measurements.
- **SHJB equation** is effectively the same thing in continuous-time, but notation becomes messy.
- The difficult thing is that we have to take into account the **future measurements**, which have not yet been observed
- Conceptually, we could **try** the following:
 - Compute an optimal estimate (e.g., posterior mean) of the state given the measurements using an **optimal filter**
 - Compute the optimal control by applying the **equations of perfectly measured state** case to the state estimate.
- However, this **assumed certainty equivalence** approach **does not lead to the optimal solution** in general.

Dual Effect of Non-Linear Stochastic Control

- **Estimation and control** and not separable in general, which leads to **dual (effect of) control**
- **Probing property:** By selecting suitable control signal, we can generate a **future measurement sequence** such that the **optimal filter performs better**. This in turn allows the controller to perform better.
- **Active adaptation/learning:** In addition to the state, we may also make the **system identification or adaptation perform better** by control selection.
- **Caution:** The presence of uncertainty also causes the optimal control to **try not to increase uncertainty** and not to make **too risky decisions**
- If we can solve a **non-linear stochastic optimal control problem**, it will **automatically** contain these properties.

Linear Quadratic Gaussian Regulator

- In **linear systems** there is **no dual effect**.
- The solution to the linear quadratic stochastic control problem is combination of **optimal filter** and **deterministic optimal controller**:
 - 1 Design a **Kalman filter**, which computes optimal estimate of the state given the measurements
 - 2 Design a **deterministic LQR** and compute the control by **feeding** it the **Kalman filter estimate** instead of the true state
- This separability of linear estimator and controller is called **certainty equivalence**.
- The combination of steady state Kalman filter and steady state LQR is called **Linear Quadratic Gaussian** (LQG) regulator

Duality of Linear Kalman-Bucy Filter and LQR

- The Kalman filter and LQR are mathematically very similar and **mathematical duals** of each other
- This has **nothing to do** with **dual effect** of control
- **Observability** of linear filtering models, **controllability** of linear systems
- **Stability** analysis, numerical methods, Riccati equation solvers

Summary [1/2]

- **Classical control:**

- Based on heuristic engineering principles for designing **linear time invariant (LTI)** controllers
- Analysis is based on **Laplace and Fourier transforms**
- **Proportional-Integral-Derivative (PID)** design is the most known controller architecture.

- **Optimal control:**

- Replaces heuristic criteria with **cost functional minimization criterion**
- Solved with **calculus of variations, dynamic programming or Hamilton-Jacobi-Bellman equation**
- **Linear Quadratic Regulator (LQR)** generalizes classical LTI controllers in the optimal control framework.

Summary [2/2]

● Stochastic control:

- Allows **uncertainties** in dynamics and measurements to be modeled using **stochastic processes**
- **Stochastic dynamic programming or stochastic Hamilton-Jacobi-Bellman equations** give the mathematical solutions to these problems.
- **Dual effect** of control means that in stochastic control the **control selection** can be used for improving the **state estimator performance** and that in turn **improves the control performance**.
- **Linear Quadratic Gaussian (LQG)** regulator is an important closed form solution to stochastic control and it is a combination of **Kalman filter** and **Deterministic Linear Quadratic Regulator**.