Introduction to Classical, Optimal and Stochastic Control

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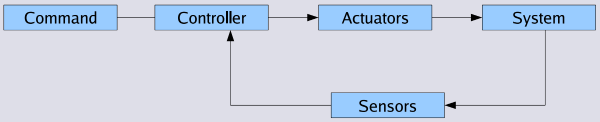
Contents

Principles and History of Control

- Control theory studies how a physical system can be steered to a given goal by applying a control signal
- Examples of applications are Cruise control, Autopilot, Spacecraft control, Chemical process control, Nuclear reactor control, Robot control and Disk drive control
- Dates back to 1700's and 1800's, but most of the classical theory was has developed in beginning of 1900's.
- The "modern" state space methods, optimal control theory, stochastic control theory and many numerical algorithms were developed in 1960's.

Building Blocks and Approaches of Control



- The physical system (plant) is monitored with sensors and the control signal is generated with actuators
- In classical control, a controller is heuristically designed such that it forces error with respect to reference to zero
- In optimal control, the controller is selected to minimize a cost functional
- In stochastic control, the system model and parameters are considered uncertain and the state is only indirectly measured

Mathematical Foundations of Control Theory

- The **physical system** is a set of differential or difference equations, deterministic or stochastic
- The **control signal** is an unknown function, which is to be determined
- In **classical control**, linear control laws are derived by heuristic methods and their stability is checked using Laplace, Fourier and Z-transforms
- In **optimal control** the control signal is solved using calculus of variations and its generalizations
- In **stochastic control**, stochastic generalizations of optimal control results are used for determining the control signals.
- Also in **stochastic control**, uncertainties and noises in the system and sensors are treated according to Bayesian probability theory

Transfer Function Models: Theory [1/2]

- The classical theory considers linear time-invariant (LTI) systems with single input and single output (SISO)
- In continuous-time, constant coefficient linear differential equations for signal $y(t)$ with input $u(t)$

$$\sum_{i=0}^{n} a_i \frac{d^i y(t)}{dt^i} = \sum_{j=0}^{m} b_j \frac{d^j u(t)}{dt^j}$$

- Analysis is based on Laplace transforms of the differential equations and their transfer functions

$$G(s) = \frac{Y(s)}{U(s)} = \frac{\sum_{j=0}^{m} b_j s^j}{\sum_{i=0}^{n} a_i s^i}$$

Transfer Function Models: Theory [2/2]

- In discrete-time, constant coefficient linear difference equations

$$\sum_{i=0}^{n} a_i\, y[t-i] = \sum_{j=0}^{m} b_j\, u[t-j]$$

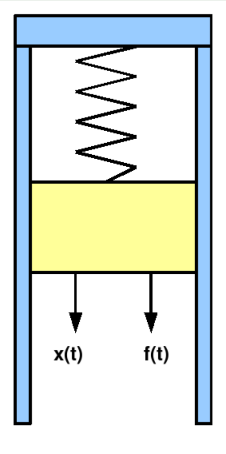
- In terms of z-transform the transfer function is

$$G(z) = \frac{Y(z)}{U(z)} = \frac{\sum_{j=0}^{m} b_j\, z^{-j}}{\sum_{i=0}^{n} a_i\, z^{-i}}$$

- Input to the system (continuous or discrete time) is the reference/command signal
- Analysis is based on impulse and step response to the input, and the steady state response

Transfer Function Models: Example [1/3]

Example



- Spring force

$$F_j = -by$$

- Friction force

$$F_k = -kdy/dt$$

- External force

$$F_f = f(t)$$

- Newton's law:

$$M\frac{d^2y}{dt^2} = F_k + F_j + F_f$$

Transfer Function Models: Example [2/3]

- If we assume M=k=1 for simplicity:

$$d^2y/dt^2 + b\,dy/dt + y = f(t)$$

- Take Laplace transform of both sides:

$$s^2Y(s) + b\,s\,Y(s) + Y(s) = F(s)$$

- This can be written in form

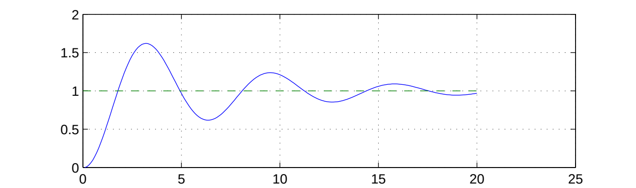
$$Y(s) = \underbrace{\left[\frac{1}{s^2 + b\,s + 1}\right]}_{G(s)} F(s)$$

- Here $G(s)$ is the transfer function from $f(t)$ to $y(t)$

Transfer Function Models: Example [3/3]

- As a block diagram the system is

- If we apply a constant force $f(t) = 1$, we get the following step response:

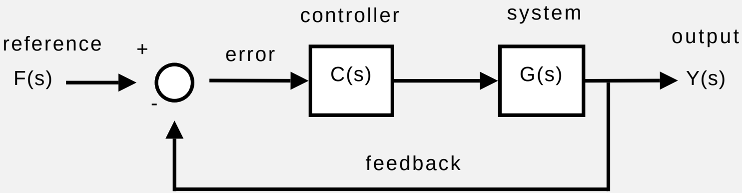


Classical Control Problem

- Classical control problem: *Design a control signal $u(t)$ such that $y(t)$ reaches its steady state as fast/well/efficiently as possible.*
- Force $f(t)$ is the reference/command signal and we want $y(t)$ to follow it
- Or might want to "measure" the force $f(t)$ by monitoring the position $y(t)$ of spring (think of accelerometer)
- How to select the controller?
- How could we get rid of the oscillations and get faster response?

Controller types: Theory

- The idea is to include controller $C(s)$ and feedback such that the system follows the given reference signal $F(s)$:



- The most known type of controller C(s) is the proportional-integral-derivative (PID) controller

$$C(s) = K_p + K_i/s + K_d\, s$$

- In time-domain, the PID-controller is

$$u(t) = K_p\, e(t) + K_i \int_0^t e(t)dt + K_d\, de/dt$$

Controller types: Example [1/3]

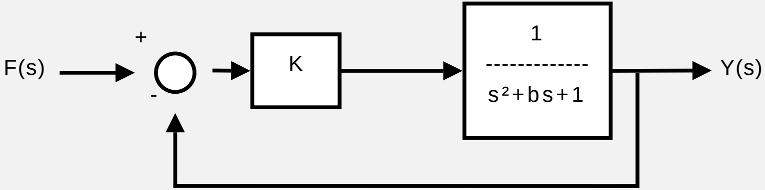
- Let's now add controller $C(s)$ to the spring system

$$Y(s) = \left[\frac{1}{s^2 + b\,s + 1}\right] F(s)$$

- Take a simple P-controller

$$C(s) = K$$

- The block diagram is the following:

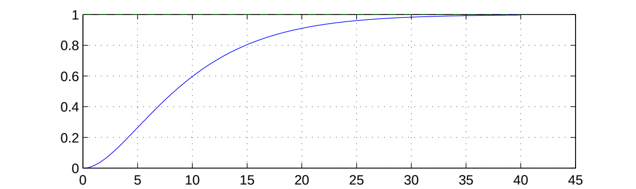


Controller types: Example [2/3]

- The closed loop system is now

$$Y(s) = \left[\frac{K}{s^2 + b\,s + 1 + K} \right] F(s)$$

- We may select K such that the system is Critically damped, that is, the fastest response before oscillations occur
- Applying a constant force $f(t) = 1$, gives the following response:



Controller types: Example [3/3]

- What this means in physical system? The equation of error $e(t) = y(t) - 1$ is now

$$u(t) = K\,e(t)$$
$$d^2e/dt^2 + b\,de/dt + e = u(t)$$

- In terms of $y(t)$

$$u(t) = K(y(t) - 1)$$
$$d^2y/dt^2 + b\,dy/dt + y = u(t)$$

- That is, we apply the following control force

$$u(t) = K(y(t) - 1)$$

Classical Design Methods

- Controller design is based on trial and error and heuristic design methods for selecting the controller parameters
- The principal design goal is stability: The system is stable when the closed loop poles are on the left-half of s-plane or inside the unit circle in z-plane
- Secondary criteria are for example, settling time, overshoot and steady state error. These can be analyzed graphically from impulse, step and ramp responses of the close loop system
- Robustness to noise and other disturbances can be analyzed using frequency domain methods
- It is also possible to design heuristic feedback laws of state space models and prove their stability, e.g., by pole placement or by Lyapunov stability theory

Problem Formulation

- The system is modeled as state space model with state vector $\mathbf{x}(t) = (x_1(t), \ldots, x_n(t))$, and control vector $\mathbf{u}(t) = (u_1(t), \ldots, u_d(t))$

$$d\mathbf{x}/dt = \mathbf{f}(\mathbf{x}, \mathbf{u}, t),$$

- The design criterion is to minimize cost functional

$$J(\mathbf{u}) = \phi(\mathbf{x}(T)) + \int_0^T L(\mathbf{x}, \mathbf{u}, t)\, dt.$$

- Initial and final states can be known, unknown or constrained
- State and control may be constrained
- The final time may be known or unknown

Euler-Lagrange and Hamiltonian Equations [1/2]

- The optimal control function $\mathbf{u}(t)$ can be derived using methods calculus of variations, namely the Euler-Lagrange equations
- The result is - we first define the Hamiltonian

$$H(\mathbf{x},\mathbf{u},\mathbf{p},t) = L(\mathbf{x},\mathbf{u},t) + \mathbf{p}^T\,\mathbf{f}(\mathbf{x},\mathbf{u},t),$$

  where $\mathbf{p}(t)$ is the co-state (= Lagrange multiplier function)
- Then the optimal control and trajectory can be solved from the Hamiltonian equations

$$\begin{aligned} d\mathbf{x}/dt &= \partial H/\partial \mathbf{p} \\ d\mathbf{p}/dt &= -\partial H/\partial \mathbf{x} \\ 0 &= \partial H/\partial \mathbf{u}, \end{aligned}$$

Euler-Lagrange and Hamiltonian Equations [2/2]

- Non-linear boundary value problem (BVP) for ordinary differential equations
- The boundary conditions depend on if the final state is known or unknown and if the final time is known or unknown
- Generally, very hard to solve even numerically
- The corresponding discrete-time optimal control problem is

$$\mathbf{x}_{k+1} = \mathbf{f}_k(\mathbf{x}_k, \mathbf{u}_k)$$

$$J(\mathbf{u}) = \sum_{k=0}^{T} L_k(\mathbf{x}_k, \mathbf{u}_k)$$

- Hamiltonian equations are almost the same, only in discrete time

Optimal Control: Example [1/2]

- Consider the spring problem:

$$d^2y/dt^2 + b\,dy/dt + y = u(t)$$

- If we define $x_1 = y$ and $x_2 = dy/dt$, this can be written as state space model

$$dx_1/dt = x_2$$
$$dx_2/dt = -x_1 - b\,x_2 + u$$

- Define the cost functional as

$$J(u) = \frac{1}{2}\int_0^T \left[||x_1(t) - f(t)||^2 + ||u(t)||^2 \right] dt$$

Optimal Control: Example [2/2]

- The Hamiltonian is now

$$H(\mathbf{x}, u, \mathbf{p}, t) = \frac{1}{2}||x_1(t) - f(t)||^2 + \frac{1}{2}||u(t)||^2 + p_1\, x_2 + p_2\, (-x_1 - b\, x_2 + u)$$

- The Hamiltonian equations now reduce to

$$\begin{aligned} dx_1/dt &= x_2 \\ dx_2/dt &= -x_1 - b\, x_2 + u \\ dp_1/dt &= -x_1 + f + p_2 \\ dp_2/dt &= -p_1 + b\, p_2 \\ 0 &= u + p_2 \end{aligned}$$

- More useful form of solution given by linear quadratic regulator framework

Pontryagin Minimum Principle

- Euler-Lagrange and Hamiltonian equations apply when control is not bounded.
- If the control is bounded, e.g., $||\mathbf{u}|| \leq 1$, then equations don't work - Pontryagin maximum (or minimum) principle must be used, where $\partial H / \partial \mathbf{u} = 0$ is replaced with

$$\mathbf{u}(t) = \min_{\mathbf{u}(t)} H(\mathbf{x}(t), \mathbf{u}(t), \mathbf{p}(t), t)$$

- Heuristically, generalizes the principle that the maximum of a function is either at point where gradient is zero or at boundary
- Especially useful in minimum time and minimum fuel problems, where there is no explicit control cost at all

Bang-Bang Control

- Minimizing the time with bounded control leads to bang-bang-control, where the control is always at the boundary, i.e., to maximum effort type of control such as

$$u(t) = \begin{cases} u_{\max} & , \quad \text{if } \varphi(\mathbf{p}(t)) < 0 \\ u_{\min} & , \quad \text{if } \varphi(\mathbf{p}(t)) > 0 \end{cases}$$

- Minimizing the fuel leads to control, where the control is always at maximum, minimum or zero
- Other constrained problems lead to solutions, which changes between bang-bang and ordinary control.
- The discrete-time version of the principle is analogous to continuous-time

Bang-Bang Control: Example [1/3]

- The equation of a forced simple pendulum is

$$d^2\theta/dt^2 = -g\,\sin(\theta) + u$$

- As state space model this is

$$\begin{aligned} dx_1/dt &= x_2 \\ dx_2/dt &= -g\,\sin(x_1) + u \end{aligned}$$

- We want to steer the pendulum from down-most position to up-most position in minimum time with bounded control $0 \le |u(t)| \le 1$
- The minimum time cost function is

$$J(u) = \int_0^T 1\, dt$$

where T is unknown.

Bang-Bang Control: Example [2/3]

- The Hamiltonian is given as

$$H(\mathbf{x}, u, \mathbf{p}) = 1 + p_1\, x_2 + p_2\,(-g\,\sin(x_1) + u)$$

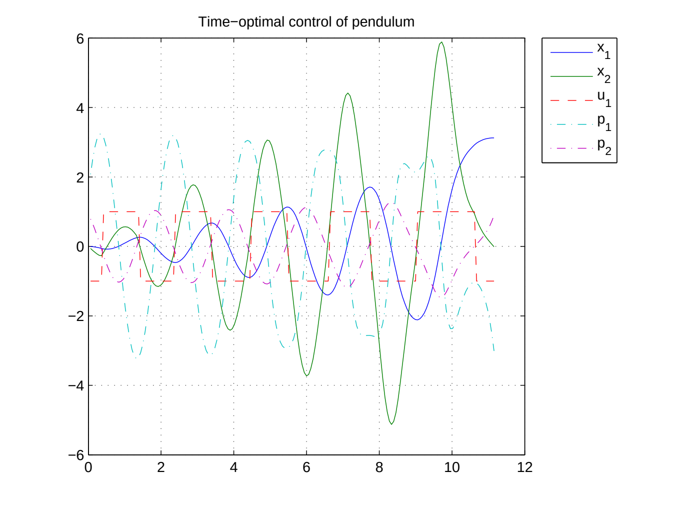
- The Hamiltonian equations are

$$\begin{aligned} dp_1/dt &= p_2\, g\, \cos(x_1) \\ dp_2/dt &= -p_1 \\ dx_1/dt &= x_2 \\ dx_2/dt &= -g\,\sin(x_1) + u \end{aligned}$$

- The Hamiltonian equation $\partial H/\partial u = 0$ does not have a nontrivial solution in the bounded domain.
- Instead, the Pontryagin principle gives:

$$u(t) = \begin{cases} 1 & , \quad \text{if } p_2(t) < 0 \\ -1 & , \quad \text{if } p_2(t) > 0 \end{cases}$$

Bang-Bang Control: Example [3/3]



Dynamic Programming [1/2]

- Bellman's Principle of Optimality: **"Starting at any point on an optimal trajectory, the remaining trajectory is optimal for the corresponding problem initiated at that point."**
- In discrete-state case this is related to finding the shortest path between two vertexes in a graph.
- We may solve the following discrete-time optimal control problem using this principle

$$\mathbf{x}_{k+1} = \mathbf{f}_k(\mathbf{x}_k, \mathbf{u}_k)$$

$$J(\mathbf{u}) = \sum_{k=0}^{T} L_k(\mathbf{x}_k, \mathbf{u}_k)$$

Dynamic Programming [2/2]

- The dynamic programming algorithm is
  1. Initialize: $V_{T+1}(\mathbf{x}_{T+1}) = 0$.
  2. For $k = T, \dots, 0$ and for all $\mathbf{x}_k$ compute

$$\mathbf{u}_k^*(\mathbf{x}_k) = \arg\min_{\mathbf{u}_k} \{ L_k(\mathbf{x}_k, \mathbf{u}_k) + V_{k+1}(\mathbf{f}_k(\mathbf{x}_k, \mathbf{u}_k)) \}$$

$$V_k(\mathbf{x}_k) = L_k(\mathbf{x}_k, \mathbf{u}_k^*) + V_{k+1}(\mathbf{f}_k(\mathbf{x}_k, \mathbf{u}_k^*)$$

  3. For $k = 0, \dots, T-1$, compute the optimal trajectory

$$\mathbf{x}_{k+1}^* = \mathbf{f}_k(\mathbf{x}_k^*, \mathbf{u}_k^*)$$

- $V_k(\mathbf{x}_k)$ is called the optimal value function or the optimal cost-to-go function
- Problem is the curse of dimensionality, because has to be solved for each $\mathbf{x}_k$ separately

Hamilton-Jacobi-Bellman Equation

- The Hamilton-Jacobi-Bellman (HJB) equation is the continuous-time limit of the dynamic programming and can be applied to models of the form

$$d\mathbf{x}/dt = \mathbf{f}(\mathbf{x}, \mathbf{u}, t)$$

$$J(\mathbf{u}) = \phi(\mathbf{x}(T)) + \int_0^T L(\mathbf{x}, \mathbf{u}, t)\, dt.$$

- The HJB equation for the optimal value function $V(\mathbf{x}, t)$ is

$$\begin{aligned} \partial V/\partial t &= -\min_{\mathbf{u}} H(\mathbf{x}, \mathbf{u}, \partial V/\partial \mathbf{x}, t) \\ &= -\min_{\mathbf{u}} \left\{ L(\mathbf{x}, \mathbf{u}, t) + (\partial V/\partial \mathbf{x})^T\ \mathbf{f}(\mathbf{x}, \mathbf{u}, t) \right\}. \end{aligned}$$

- Non-linear partial differential equation and very hard to solve even in simple cases
- Gives the optimal control in state feedback form $\mathbf{u}^*(\mathbf{x}(t), t)$

Linear Quadratic Regulator

- The linear quadratic regulator (LQR) model is

$$
\begin{aligned}
\mathrm{d}\mathbf{x}(t)/\mathrm{d}t &= \mathbf{F}(t)\,\mathbf{x}(t) + \mathbf{B}(t)\,\mathbf{u}(t) \\
J(\mathbf{u}) &= \frac{1}{2}\mathbf{x}(T)^T\,\boldsymbol{\Phi}\,\mathbf{x}(T) \\
&\quad + \frac{1}{2}\int_0^T \left(\mathbf{x}^T(t)\,\mathbf{Q}(t)\,\mathbf{x}(t) + \mathbf{u}^T(t)\,\mathbf{R}(t)\,\mathbf{u}(t)\right)\mathrm{d}t,
\end{aligned}
$$

- The HJB equation now has a closed form solution:

$$
\begin{aligned}
V(\mathbf{x}(t), t) &= \frac{1}{2}\mathbf{x}^T(t)\,\mathbf{K}(t)\,\mathbf{x}(t) \\
\mathbf{u}(t) &= -\mathbf{R}^{-1}(t)\,\mathbf{B}^T(t)\,\mathbf{K}(t)\,\mathbf{x}(t)
\end{aligned}
$$

- The matrix $\mathbf{K}(t)$ solves the Riccati differential equation

$$
\begin{aligned}
\mathrm{d}\mathbf{K}(t)/\mathrm{d}t &= -\mathbf{F}^T(t)\,\mathbf{K}(t) - \mathbf{K}(t)\,\mathbf{F}(t) \\
&\quad + \mathbf{K}(t)\,\mathbf{B}(t)\,\mathbf{R}^{-1}(t)\,\mathbf{B}^T(t)\,\mathbf{K}(t) - \mathbf{Q}(t).
\end{aligned}
$$

Steady State Linear Quadratic Regulator

- Consider LTI linear quadratic regulator model, where the dynamic model is LTI, the terminal time is at infinity and there is no terminal cost:

$$\mathrm{d}\mathbf{x}(t)/\mathrm{d}t = \mathbf{F}\,\mathbf{x}(t) + \mathbf{B}\,\mathbf{u}(t)$$

$$J(\mathbf{u}) = \frac{1}{2}\int_0^\infty (\mathbf{x}^T(t)\,\mathbf{Q}\,\mathbf{x}(t) + \mathbf{u}^T(t)\,\mathbf{R}\,\mathbf{u}(t))\,\mathrm{d}t,$$

- The solution is now time invariant:

$$V(\mathbf{x}(t), t) = \frac{1}{2}\mathbf{x}^T(t)\,\mathbf{K}\,\mathbf{x}(t)$$

$$\mathbf{u}(t) = -\mathbf{R}^{-1}\,\mathbf{B}^T\,\mathbf{K}\,\mathbf{x}(t)$$

- The constant matrix **K** solves the algebraic Riccati equation

$$0 = -\mathbf{F}^T\,\mathbf{K} - \mathbf{K}\,\mathbf{F} + \mathbf{K}\,\mathbf{B}\,\mathbf{R}^{-1}\,\mathbf{B}^T\,\mathbf{K} - \mathbf{Q}.$$

Linear Quadratic Regulator: Example [1/3]

- The error equation of the spring control problem:

$$d^2e/dt^2 + b\,de/dt + e = u(t)$$

- If we define $x_1 = e$ and $x_2 = de/dt$, can be written as linear state space model

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & -b \end{pmatrix}}_{\mathbf{F}} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\mathbf{B}} u$$

- Define the cost functional as

$$J(u) = \frac{1}{2}\int_0^\infty \left[||e(t)||^2 + c\,||u(t)||^2 \right] dt$$

Linear Quadratic Regulator: Example [2/3]

- The elements of 2×2 matrix $\mathbf{K}$ can be solved from the algebraic equations

$$
\begin{aligned}
k_{21} + k_{12} + k_{12}\, k_{21}/c - 1 &= 0 \\
k_{22} - k_{11} + b\, k_{12} + k_{12}\, k_{22}/c &= 0 \\
-k_{11} + b\, k_{21} + k_{22} + k_{22}\, k_{21}/c &= 0 \\
-k_{12} + 2b\, k_{22} - k_{21} + k_{22}^2/c - 1 &= 0
\end{aligned}
$$

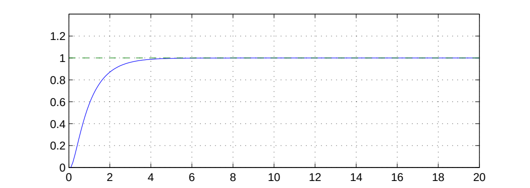
- The optimal control is of the form

$$
u(t) = -\mathbf{B}^T\, \mathbf{K}\, \mathbf{x}(t) = -k_{21}\, e(t) - k_{22}\, de/dt
$$

- This is a proportional-derivative (PD) controller

Linear Quadratic Regulator: Example [3/3]

- The step response of the PD controller is



- If we had augmented the system with integral state, we would have obtained a PID controller
- All classical controllers have a counterpart in LQR theory
- LQR framework provides a systematic way of selecting the PID parameters instead of heuristics

Problem Formulation - Perfectly Measured State

- The system contains uncertainties in dynamics, which are modeled with noise process $\mathbf{w}(t)$:

$$\mathrm{d}\mathbf{x}/\mathrm{d}t = \mathbf{f}(\mathbf{x}, \mathbf{u}, t) + \mathbf{L}(t)\,\mathbf{w}(t)$$

- The performance criterion is the expected value of the cost function:

$$J(\mathbf{u}) = \mathrm{E}\left[\phi(\mathbf{x}(T)) + \int_0^T L(\mathbf{x}, \mathbf{u}, t)\,\mathrm{d}t\right].$$

- In discrete-time case the corresponding model is:

$$\mathbf{x}_{k+1} = \mathbf{f}_k(\mathbf{x}_k, \mathbf{u}_k, \mathbf{w}_k)$$

$$J(\mathbf{u}) = \mathrm{E}\left[\sum_{k=0}^{T} L_k(\mathbf{x}_k, \mathbf{u}_k)\right]$$

- The state is assumed to be completely known when the control is applied

Stochastic Dynamic Programming and HJB I

- The discrete-time solution is almost the same as in deterministic case, with a few additional expected values:

$$\mathbf{u}_k^*(\mathbf{x}_k) = \arg\min_{\mathbf{u}_k} \mathrm{E}_{w_k}\left[L_k(\mathbf{x}_k, \mathbf{u}_k) + V_{k+1}(\mathbf{f}_k(\mathbf{x}_k, \mathbf{u}_k, \mathbf{w}_k))\right]$$
$$V_k(\mathbf{x}_k) = \mathrm{E}_{w_k}\left[L_k(\mathbf{x}_k, \mathbf{u}_k^*) + V_{k+1}(\mathbf{f}_k(\mathbf{x}_k, \mathbf{u}_k^*, \mathbf{w}_k)\right]$$

- In continuous-time case there are no expected values, but the stochastic HJB equation contains an additional second order term:

$$\partial V / \partial t = -\min_{\mathbf{u}} \Bigg\{ L(\mathbf{x}, \mathbf{u}, t) + (\partial V / \partial \mathbf{x})^T \ \mathbf{f}(\mathbf{x}, \mathbf{u}, t) + \frac{1}{2} \operatorname{tr}\left[\left(\frac{\partial^2 V}{\partial \mathbf{x}^2}\right) \mathbf{L}(t)\, \mathbf{Q}_c(t)\, \mathbf{L}^T(t)\right] \Bigg\}$$

Stochastic Linear Quadratic Regulator

- The stochastic linear quadratic regulator (LQR) model is

$$
\begin{aligned}
\mathrm{d}\mathbf{x}(t)/\mathrm{d}t &= \mathbf{F}(t)\,\mathbf{x}(t) + \mathbf{B}(t)\,\mathbf{u}(t) + \mathbf{L}(t)\,\mathbf{w}(t) \\
J(\mathbf{u}) &= \mathrm{E}\left[\frac{1}{2}\mathbf{x}(T)^T\,\boldsymbol{\Phi}\,\mathbf{x}(T)\right. \\
&\quad \left. + \frac{1}{2}\int_0^T \left(\mathbf{x}^T(t)\,\mathbf{Q}(t)\,\mathbf{x}(t) + \mathbf{u}^T(t)\,\mathbf{R}(t)\,\mathbf{u}(t)\right)\mathrm{d}t\right]
\end{aligned}
$$

- The optimal control is exactly the same as in LQR without the noise $\mathbf{w}(t)$.
- The optimal value function is of the form

$$
V(\mathbf{x}(t), t) = V_d(\mathbf{x}(t), t) + g(t)
$$

where $V_d(\cdot)$ is the optimal value function of deterministic case.

Problem Formulation - Partially Measured State [1/2]

- In physical systems the state of the system is never perfectly known, but we observe it through some sensors
- Model for sensor measurement $\mathbf{y}(t)$ with noise/uncertainties $\mathbf{v}(t)$:

$$\mathbf{y}(t) = \mathbf{h}(\mathbf{x}(t), \mathbf{u}(t), t) + \mathbf{v}(t)$$

- We do not observe $\mathbf{x}(t)$, only $\mathbf{y}(t)$ and all our decisions must be based on the already observed $\mathbf{y}(t)$
- The control decision at time t is function of measurements obtained before and at time t
- If the control $\mathbf{u}(t)$ was given, the estimation of $\mathbf{x}(t)$ given also the dynamic model would be an optimal filtering problem.
- The situation is much more complicated - aside with optimal filtering, we want to also optimize the expected value of cost function.

Problem Formulation - Partially Measured State [2/2]

- The whole model in continuous time:

$$\begin{aligned} \mathrm{d}\mathbf{x}/\mathrm{d}t &= \mathbf{f}(\mathbf{x}, \mathbf{u}, t) + \mathbf{L}(t)\,\mathbf{w}(t) \\ \mathbf{y} &= \mathbf{h}(\mathbf{x}, \mathbf{u}, t) + \mathbf{v}(t) \\ J(\mathbf{u}) &= \mathrm{E}_x \left[\phi(\mathbf{x}(T)) + \int_0^T L(\mathbf{x}, \mathbf{u}, t)\,\mathrm{d}t \,\middle|\, Y_t \bigcup Y^{+}_{[t,T]} \right]. \end{aligned}$$

- In discrete time:

$$\begin{aligned} \mathbf{x}_{k+1} &= \mathbf{f}_k(\mathbf{x}_k, \mathbf{u}_k, \mathbf{w}_k) \\ \mathbf{y}_k &= \mathbf{h}_k(\mathbf{x}_k, \mathbf{u}_k, \mathbf{v}_k) \\ J(\mathbf{u}) &= \sum_{k=0}^{T} \mathrm{E}_x \left[L_k(\mathbf{x}_k, \mathbf{u}_k) \,\middle|\, \mathbf{y}_{1:k} \bigcup \mathbf{y}^{+}_{k+1:T} \right] \end{aligned}$$

Form of the Solution

- The discrete-time solution resembles the perfectly measured case, but the value function is now function of the measurements.
- SHJB equation is effectively the same thing in continuous-time, but notation becomes messy.
- The difficult thing is that we have to take into account the future measurements, which have not yet been observed
- Conceptually, we could try the following:
  - Compute an optimal estimate (e.g., posterior mean) of the state given the measurements using an optimal filter
  - Compute the optimal control by applying the equations of perfectly measured state case to the state estimate.
- However, this assumed certainty equivalence approach does not lead to the optimal solution in general.

Dual Effect of Non-Linear Stochastic Control

- Estimation and control and not separable in general, which leads to dual (effect of) control
- **Probing property:** By selecting suitable control signal, we can generate a future measurement sequence such that the optimal filter performs better. This in turn allows the controller to perform better.
- **Active adaptation/learning**: In addition to the state, we may also make the system identification or adaptation perform better by control selection.
- **Caution:** The presence of uncertainty also causes the optimal control to try not to increase uncertainty and not to make too risky decisions
- If we can solve a non-linear stochastic optimal control problem, it will automatically contain these properties.

Linear Quadratic Gaussian Regulator

- In linear systems there is no dual effect.
- The solution to the linear quadratic stochastic control problem is combination of optimal filter and deterministic optimal controller:
  1. Design a Kalman filter, which computes optimal estimate of the state given the measurements
  2. Design a deterministic LQR and compute the control by feeding it the Kalman filter estimate instead of the true state
- This separability of linear estimator and controller is called certainty equivalence.
- The combination of steady state Kalman filter and steady state LQR is called Linear Quadratic Gaussian (LQG) regulator

Duality of Linear Kalman-Bucy Filter and LQR

- The Kalman filter and LQR are mathematically very similar and mathematical duals of each other
- This has nothing to do with dual effect of control
- Observability of linear filtering models, controllability of linear systems
- Stability analysis, numerical methods, Riccati equation solvers

Summary [1/2]

- **Classical control**:
  - Based on heuristic engineering principles for designing linear time invariant (LTI) controllers
  - Analysis is based on Laplace and Fourier transforms
  - Proportional-Integral-Derivative (PID) design is the most known controller architecture.
- **Optimal control**:
  - Replaces heuristic criteria with cost functional minimization criterion
  - Solved with calculus of variations, dynamic programming or Hamilton-Jacobi-Bellman equation
  - Linear Quadratic Regulator (LQR) generalizes classical LTI controllers in the optimal control framework.

Summary [2/2]

- **Stochastic control**:
  - Allows uncertainties in dynamics and measurements to be modeled using stochastic processes
  - Stochastic dynamic programming or stochastic Hamilton-Jacobi-Bellman equations give the mathematical solutions to these problems.
  - Dual effect of control means that in stochastic control the control selection can be used for improving the state estimator performance and that in turn improves the control performance.
  - Linear Quadratic Gaussian (LQG) regulator is an important closed form solution to stochastic control and it is a combination of Kalman filter and Deterministic Linear Quadratic Regulator.