Introduction to Bayesian Estimation of Time-Varying Processes

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4 Summary
Estimation of Dynamic Process

- **Dynamic**, that is, time varying phenomenon - for example the motion state of a car.
- The phenomenon is measured - for example by a radar or by acceleration and angular velocity sensors.
- The purpose is to compute the state of the phenomenon when only the measurements are observed.
- The solution should be recursive, where the information in new measurements is used for updating the old information.
The laws of physics are typically differential equations, as well as models of biological, epidemiological, chemical and many other phenomena.

Uncertainties and unknown sub-phenomena are modeled as stochastic processes:

- Continuous-time phenomena (e.g., movement of bodies) can be modeled as stochastic differential equations (continuous-time Markov processes).
- Discrete-time phenomena (e.g., digital signals) can be modeled as stochastic difference equations (discrete-time Markov processes).
The relationship between measurements and phenomenon is mathematically modeled as a probability distribution.

The measurements could be (in ideal world) computed if the phenomenon was known.

The uncertainties in measurements and model are modeled as random processes.

The measurement model is the conditional distribution of measurements given the state of the phenomenon.
The mathematical solution to the problem is called the **optimal filter**.

- The optimal filter computes the optimal, that is, the **minimum error estimate** of the state of the phenomenon.
- The solution is called **filter** for historical reasons.
- The mathematical equations can be derived from the theory of **Bayesian inference** and theory of **stochastic differential/difference equations**.
- In practice, the equations have to be solved numerically ⇒ different kinds of optimal filters.
In **discrete-time filtering** both the state and measurements are modeled as **discrete-time processes**.

In **continuous-discrete (-time) filtering** the state dynamics are modeled as a **continuous-time process** and the measurements are modeled as a **discrete-time process**.

In **continuous-time filtering** both the state dynamics and measurements are modeled as **continuous-time processes**.
Kalman filter is the classical optimal filter for linear-Gaussian models.

Extended Kalman filter (EKF) is linearization based extension of Kalman filter to non-linear models.

Unscented Kalman filter (UKF) is sigma-point transformation based extension of Kalman filter.

Particle filter forms a Monte Carlo representation (particle set) to the distribution of the state estimate.

Grid based filters approximate the probability distributions by a finite grid.

Mixture Gaussian approximations are used, for example, in multiple model Kalman filters and Rao-Blackwellized Particle filters.
The navigation system of Eagle lunar module AGC was based on an optimal filter - this was in the year 1969.

The dynamic model was Newton’s gravitation law.

The measurements at lunar landing were the radar readings.

On rendezvous with the command ship the orientation was computed with gyroscopes and their biases were also compensated with the radar.

The optimal filter was an extended Kalman filter.
The dynamic model in GPS receivers is often the Newton’s second law where the force is completely random, that is, the *Wiener velocity model*. The measurements are *time delays of satellite signals*. The optimal filter computes the position and the accurate time. Also the errors caused by *multi path* can be modeled and compensated. *Acceleration and angular velocity measurements* are sometimes used as extra measurements.
Spreading of a disease in population can be modeled by differential equations.

Modeling the unknown parameters and phenomena as random processes leads to stochastic dynamic model.

The measurements in this case are the number of dead or recovered individuals.

Optimal filter is used for computing the unknown parameters and the number of susceptible and infected individuals.

It is also possible to predict when the maximum of the epidemic and how many casualties there will be.
Other Applications

- **Target tracking**, where one or many targets are tracked with many passive sensors - *air surveillance*.
- **Time series prediction** where the model parameters are estimated from the measured time series and the unknown part is predicted using these parameters.
- **Analysis and restoration of audio signals**.
- **Telecommunication systems** - optimal receivers, signal detectors.
- **State estimation of control systems** - auto pilots, robot control, control systems of cars.
In optimal filtering the state of a dynamic, that is, time varying phenomenon is computed from measurements obtained from it.

Uncertainties are modeled as random variables or random processes.

Numerical methods of optimal filtering are, for example, Kalman filters and particle filters.

The methods have applications in, for example, navigation, estimation of spread of disease, time series analysis, audio signal analysis, telecommunications and control systems.