# Overview of Image Analysis Methods and the Wiener Filter

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# Image as Mathematical Object

- Matrix of gray values, where $u, v \in \mathbb{N}$ are indices:

$$F = \begin{pmatrix} g_{11} & \cdots & g_{W1} \\ \vdots & \ddots & \\ g_{1H} & & g_{WH} \end{pmatrix}$$

- Function $f : \mathbb{R}^2 \mapsto \mathbb{R}$:

$$f(u, v) = \text{intensity value at } (u, v)$$

- Matrix valued random variable with probability distribution $p(F)$
- Spatial stochastic process or random field $f$ indexed by $(u, v) \in \mathbb{R}^2$:

$$F(\omega) = \{f(u, v; \omega) \ : \ (u, v) \in \mathbb{R}^2\},$$

# Image Processing/Analysis Tasks

- Noise/degradation reduction, image enhancement:
  - *Statistical estimation problem*
  - *Variational / optimization problem*
  - *Solution of partial differential equation (PDE)*
- Correction of image distortions, image restoration
  - *Statistical, variational or PDE problem*
- Image segmentation and object detection
  - *Statistical estimation problem*
  - *Discrete optimization problem*
- Image analysis and understanding
  - *Bayesian decision theory (statistical estimation)*
  - *Optimization problem*

# Models and Methods

- Basic image processing methods
  - *Convolution filters*
  - *Histogram processing and thresholding*
  - *Image segmentation and edge detection*
  - *Feature based object classification*
  - *Morphological operations*
- Stochastic and PDE models
  - *Gaussian processes, Wiener filter*
  - *Variational optimization models*
  - *Partial differential equation models*
  - *Random field models*
  - *Autoregressive texture models*
  - *State space models*

# Image Processing/Analysis Models and Methods (cont.)

- Transform based methods
  - *Fourier/Hadamard transforms*
  - *Radon and Hough transforms*
  - *Wavelet methods*
- Other methods
  - *Differential geometric methods*
  - *Object size and shape models*
  - *Non-linear filtering*
  - . . . and many more

# Connections to Other Fields

- Gaussian process regression
  - *Wiener filter is Gaussian process regressor*
  - *Reproducing kernel Hilbert space (RKHS) formulation of GPs*
- Physics
  - *Meteorology, spatial models, stochastic PDE models*
  - *Diffusion equation, fluid mechanics, statistical physics*
  - *Hamilton/Jacobi/Lagrange equations, quantum mechanics*
- Control theory
  - *Distributed parameter models*
  - *Optimal estimation, Kalman filters and Wiener filters*
  - *Optimal control as variational problem*

# Image as Gaussian Process

- The image is a Gaussian process with given covariance function

$$E[f(u,v)\, f^T(\tilde{u},\tilde{v})] = C_{ff}(u,v,\tilde{u},\tilde{v})$$

- The observed image $y$ is corrupted by additive Gaussian noise process $n$

$$y(u,v) = f(u,v) + n(u,v)$$

- The noise has known covariance function

$$E[n(u,v)\, n^T(\tilde{u},\tilde{v})] = C_{nn}(u,v,\tilde{u},\tilde{v})$$

- Generalizations: Measurement model can include linear operator, be observed in discrete points etc.

# Wiener-Hopf Equation

- The posterior mean has the representation

$$m(u, v) = \int \int h(u, v, \tilde{u}, \tilde{v})\, y(\tilde{u}, \tilde{v})\, d\tilde{u}\, d\tilde{v}$$

- The point spread function $h$ given by Wiener-Hopf equation

$$\begin{aligned} C_{ff}(u, v, \tilde{u}, \tilde{v}) = \int \int & h(u, v, \tilde{u}', \tilde{v}')\, [C_{ff}(\tilde{u}', \tilde{v}', \tilde{u}, \tilde{v}) \\ & + C_{nn}(\tilde{u}', \tilde{v}', \tilde{u}, \tilde{v})]\, d\tilde{u}'\, d\tilde{v}' \end{aligned}$$

- Operator analogs to the matrix equations

$$\begin{aligned} m &= H\, y \\ C_{ff} &= H\, (C_{ff} + C_{nn}) \end{aligned}$$

- These matrix equations can be solved:

$$m = C_{ff}\, (C_{ff} + C_{nn})^{-1}\, y$$

# Wiener Filter as Convolution Filter

- The point spread function $h$ is the Wiener filter
- Problem is simplified if we assume homogeneity:

$$E[f(u,v)\,f^T(u+\tilde{u},v+\tilde{v})] = C_{ff}(\tilde{u},\tilde{v})$$
$$E[n(u,v)\,n^T(u+\tilde{u},v+\tilde{v})] = C_{nn}(\tilde{u},\tilde{v})$$

- The Wiener-Hopf reduces to convolution

$$C_{ff}(u,v) = \int\int h(u-\tilde{u},v-\tilde{v})\,[C_{ff}(\tilde{u},\tilde{v}) + C_{nn}(\tilde{u},\tilde{v})]\,d\tilde{u}\,d\tilde{v}$$

- The kernel $h$ is translation invariant and Wiener filter is a convolution filter

$$m(u,v) = \int\int h(u-\tilde{u},v-\tilde{v})\,y(\tilde{u},\tilde{v})\,d\tilde{u}\,d\tilde{v}$$

# Wiener Filter in Spectral Domain

- The Fourier transforms of homogeneous covariance functions give the spectral densities:

$$S_{ff}(\omega_u, \omega_v) = F[C_{ff}(u, v)]$$
$$S_{nn}(\omega_u, \omega_v) = F[C_{nn}(u, v)]$$

- Due to convolution theorem, the Wiener-Hopf reduces to

$$S_{ff}(\omega_u, \omega_v) = H(\omega_u, \omega_v)\,[S_{ff}(\omega_u, \omega_v) + S_{nn}(\omega_u, \omega_v)]$$

- Solving for Fourier transform of $h$ gives the transfer function

$$H(\omega_u, \omega_v) = \frac{S_{ff}(\omega_u, \omega_v)}{S_{ff}(\omega_u, \omega_v) + S_{nn}(\omega_u, \omega_v)}$$

- The filtering operation in terms of Fourier transforms of measurement $Y$ and estimate $M$ is then

$$M(\omega_u, \omega_v) = H(\omega_u, \omega_v)\,Y(\omega_u, \omega_v)$$

# Stochastic Partial Differential Equation Models

- Physics based models can be sometimes formulated as *stochastic partial differential equations* of the form

$$L[f(u,v)] = w(u,v)$$

  where $L[]$ is partial differential operator and $w(u,v)$ white Gaussian random field
- For example, stochastic Poisson equation:

$$\frac{\partial^2 f}{\partial^2 u} + \frac{\partial^2 f}{\partial^2 v} = w(u,v)$$

- If $G$ is the Green's function of operator $L[]$ and white noise has spectral density $q$, then $f$ is a Gaussian process with covariance

$$C_{ff}(u,v,\tilde{u},\tilde{v}) = q \int\int G(u,v,\tilde{u}',\tilde{v}')\, G(\tilde{u}',\tilde{v}',\tilde{u},\tilde{v})\, d\tilde{u}'\, d\tilde{v}'$$

# Variational Optimization and RKHS Problems

- Consider the following SPDE model:

$$\begin{aligned} L[f(u,v)] &= w(u,v) \\ y(u,v) &= f(u,v) + n(u,v) \end{aligned}$$

- Could be solved by transforming into Wiener filtering or Gaussian process regression problem
- Alternatively - the posterior mean is the solution to the variational minimization of

$$J[m] = \int\int [(y(u,v) - m(u,v))^2 + \lambda |L[m]|^2] \, du \, dv$$

- Connected to Tikhonov regularization and spline regression

# Reproducing Kernel Hilbert Space (RKHS) Method

- In RKHS the regularizer $|L[m]|^2$ is replaced by special inner product $< \cdot, \cdot >_H$:

$$J[m] = \int \int (y(u,v) - m(u,v))^2 \, du \, dv + \lambda < m, m >_H$$

- The inner product is defined such that it has the reproducing property with respect to covariance $C_{ff}$:

$$< C_{ff}(u,v,*), f(*) >_H = f(u,v)$$

- Now it is possible to solve the variational problem without knowing (or existence) or the differential operator $L[]$

# Partial Differential Equation Models

- The variational minimization of $J[m]$ leads to partial differential equation of the form

$$m + \lambda \, L^*[L[m]] = y$$

- The Wiener filter is the Fourier transform solution to this equation
- The equation can be also solved with finite elements (FEM), finite differences (FD) or other numerical methods.
- PDE models for images can be also directly constructed by applying physical or invariance principles

# Summary

- Overview of image analysis methods and connections to other fields have been presented
- Wiener filter solutions to Gaussian process inference problem have been presented:
  - The spatial domain solution is an integral operator
  - For homogeneous processes the solution is a convolution operation or spectral domain multiplication
- The connection to variational and partial differential equation methods has been illustrated