Problem 1. (pencil and paper / computer) (2 points)

(a) Suppose that you use the Monte Carlo method to estimate the value of a given quantity $A$. One simulation run gives you a single numerical value denoted by $A_i$. How can you obtain a reliable estimate of the true value of $A$ and how can you calculate an estimate of the error?

(b) Write a program that computes an estimate of $\pi$ using the "hit-or-miss" Monte Carlo method. The program should also calculate an error estimate $\sigma$ (as you described in part (a)) and average deviation from the correct value ($\pi = 3.141592654$).

Do this by generating $N$ uniformly distributed random points inside the square defined by $-R \leq x \leq R$ and $-R \leq y \leq R$. Calculate the number of points which are inside the circle $x^2 + y^2 = R^2$. For each value of $N$, perform $n = 1000$ independent measurements. The resulting average value

$$\pi_{est} = \langle \pi \rangle = \frac{1}{n} \sum_{i=0}^{n} \pi_i$$

is your MC estimate. The average deviation from the correct value is given by

$$\frac{1}{n} \sum_{i=0}^{n} |\pi_i - \pi|$$

Plot your estimate of $\pi$, the error estimate $\sigma$ and the average absolute error as a function of $N$ for $N = 1000 - 30000$ (e.g. do a series of runs increasing $N$ by 1000 at each round). What relation does the error estimate follow? What about the absolute error?
Problem 2. (computer) 
Write a program which uses the "sample-mean" method to compute the integral

\[ I = \int_{0}^{2} \left\{ \int_{3}^{6} \left[ \int_{-1}^{1} (yx^2 + z\ln y + e^x)dx \right] dy \right\} dz \]

Include the calculation of an error estimate in your program. Show the results as a function of \( N \) (number of random points). The correct answer is \( I \approx 49.9213 \).

Problem 3 (computer) 
(a) Write a program which uses importance sampling to compute the integral

\[ I = \int_{1}^{2} (2x^8 - 1)dx \]

Plot the result as a function of \( N \).

(b) Compute the same integral using the sample mean method. Plot the absolute error \( |I_{est} - I| \) as a function of \( N \) for the results obtained using the sample mean method and for the results obtained in (a) using importance sampling. Which method is better?

Problem 4. (computer) 
Write a program which simulates the process of radioactive decay. You are given a sample of \( N = 20000 \) radioactive nuclei each of which decays at a rate \( p \) per second. What is the half-life of the sample if \( p = 0.4 \)?

Calculate the estimate of the half-life \( \langle t_{1/2} \rangle \) from \( m \) independent measurements and include an estimation of the error in your program. Increase \( m \) until you reach an accuracy of at least 0.005.