**S-114.100 Computational Science / Laskennallinen tiede. Fall 2004.**

Assignment 8. Random numbers
Lecture notes 8.
Due Thu 17.11.2004 (4 problems, total of 5 points)

Web page: `www.lce.hut.fi/teaching/S-114.100/`
Codes: `www.lce.hut.fi/teaching/S-114.100/codes/`

*computer* = programming task (C / C++ / Fortran / Java)
*pencil and paper* = solve on paper

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**Problem 1.** (*computer*)
The oldest random number generators are linear congruential generators (LCG),

$$x_{i+1} = (ax_i + b) \bmod m$$

where the numbers $x_i$ give the sequence of random integers. Uniform random numbers in [0,1] are obtained by scaling the $x_i$'s. The essential parameters are $a$, $b$ and $m$, and typically the notation LCG(a,b,m) is used.

Program the random number generator LCG(128,0,509). Use the program to plot 10000 points in the square $x, y \in [0,1]$ given by consecutive random number pairs $x = x_i$ and $y = x_{i+1}$. Do the same with the generator RANMAR (code can be downloaded from the course web page). Interpret the results.

**Problem 2.**
**(a)** (*pencil and paper*)
Device a method for generating uniformly distributed random points inside the sphere defined by $x^2 + y^2 + z^2 = R^2$. Use the general method of transforming probability densities. (You have a random number generator that produces uniformly distributed random numbers in $[0,1]$).

*Hint:* The cumulative distribution function in three dimensions is

$$F(r,\theta,\phi) = \frac{3}{4\pi R}\int_0^r \int_0^\theta \int_0^\phi r'^2 \sin\theta' dr' d\theta' d\phi'$$

where $0 \le r \le R$, $0 \le \theta \le \pi$ and $0 \le \phi \le 2\pi$. The random number generator gives three random variables $s_1$, $s_2$ and $s_3$, uniformly distributed between 0 and 1. Their cumulative distribution is

$$G(s_1,s_2,s_3) = \int_0^{s_1} \int_0^{s_2} \int_0^{s_3} ds_1' ds_2' ds_3'$$

In order to transform $s_1$ into $r$, $s_2$ into $\theta$ and $s_3$ into $\phi$, equate the respective integrals in the two distribution functions $F$ and $G$. This gives you three formulas for obtaining $r$, $\theta$ and $\phi$ using the three random number variables $s_1$, $s_2$ and $s_3$. Finally, the transformation between spherical and cartesian coordinates is given by

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

**Problem 2.** (continued)
**(b)** (*computer*)
Write a program that uses this method to generate 1600 random points uniformly distributed in the sphere defined by $x^2+y^2+z^2=1$. Count the number of points in the first octant. (Use RANMAR to generate the uniformly distributed random numbers).

**Problem 3** (*computer*)
Using the Box-Muller algorithm, write a program that produces Gaussian distributed random numbers. Use the program to generate $10^5$ Gaussian random numbers $y_i$. Plot the resulting distribution by dividing the interval $[-5,5]$ into 100 bins (subintervals) and by counting the number of $y_i$'s which fall into each bin. Does the distribution have a Gaussian shape?

*Hint:* In order to count the histogram (= number of $y_i$'s in each bin), you must assign each random number to a correct bin. The bin for a given random number $y_i$ is given by

$$\text{bin} = [(y_i - y_{min})/(y_{max} - y_{min})] * N_{bin}$$

where $y_{min}$ and $y_{max}$ are the minimum and maximum values of the generated random numbers and $N_{bin}$ is the number of bins.

**Problem 4.** (*computer*) (2 points)
**(a)** Using $N$ random numbers, the $k$th moment of the distribution can be estimated by

$$\langle x^k \rangle = \frac{1}{N}\sum_{i=1}^{N} x_i^k$$

Test the generators LCG(128,0,509) and RANMAR by calculating the first three moments $\langle x \rangle$, $\langle x^2 \rangle$ and $\langle x^3 \rangle$ as a function of $N$. Compare to the exact value which for the uniform distribution is $\langle x^k \rangle = 1/(k+1)$.

**(b)** Main idea of the central limit theorem is: *An average of measured quantities, taken from the same distribution (which can be any distribution), will asymptotically follow the normal distribution*. The larger the number of measurements, the closer to normal the distribution is.

Check if this applies for the values of $\langle x^2 \rangle$ obtained using RANMAR and taking an average over $N = 100$ random numbers. Take $m = 10000$ independent measurements of $\langle x^2 \rangle$, and divide the values into 100 bins in the interval [0.2,0.5]. Plot the resulting histogram. Does it resemble the normal distribution?